Strategyproof location mechanisms in graphs

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1. The context

Social choice theory offered to science and society two remarkable(y) negative results. First, by the celebrated Arrow's impossibility theorem [3], the only voting function satisfying natural fairness axioms is a dictatorial one. On the other hand, by the classical Gibbard–Satterthwaite theorem [15, 24] any voting function is manipulable unless it is dictatorial. Arrow's theorem together with Condorcet's paradox in voting theory motivated the investigation of consensus functions on general domains and to an axiomatic characterization of some important consensus functions. On the other hand, the Gibbard–Satterthwaite phenomenon lead to the concepts of *strategyproofness* and *truthful mechanisms*, which were defined and intensively investigated in voting theory [8, 9, 21] as well as in other research areas ranging from electronic commerce (Internet-related optimization problems) [14, 23], learning and classification [13, 20], operation research [1, 14, 16, 25], and algorithmic game theory [1, 14, 16, 18, 19, 23]. In these settings, given is a set of selfish agents, each endowed with its own cost (or utility) function and a *mechanism* (or a *consensus function*) is a function that selects an outcome that optimize some collective objective function. Given the outcome, each agent can compute the value of his cost function for this outcome. The mechanism is called *strategyproof* (or *truthful*) if no agent is able to manipulate the choice rule and improve his individual cost by misreporting his individual choice. Such social choice mechanisms do not utilize payments.

A particularly nice setting was described by Schummer and Vohra [25] and concerns the facility location problems. In this setting, each agent i selects his preferred location in some network, graph, or general metric space (X, d) (i.e., a point $x_i \in X$) and his individual cost function is simply the distance from x_i to points of X. The output for this location profile $(x_1, \ldots, x_n)$ is the location $x \in X$ of a facility and the objective is to minimize either the *social cost* (the sum of agent's distances to the facility) or the maximum cost (the largest distance to the facility). The point minimizing the first function is the classical *median point* and the problem of computing such a point is called the *median problem*. The point minimizing the second function is called the *center point* and the problem of computing such a point is called the *center problem*. The strategyproofness in this case means that no agent i can improve his distance to the facility by misreporting his position x_i (i.e., reporting x'_i instead of x_i). We call this a *strategyproof location mechanism* (or a *SP location mechanism*).

There are two types of problems concerning SP location mechanisms. First, proving that the mechanism returning for each location profile a median point or a center point is strategyproof. Second, if the median or the center is not strategyproof, then algorithmically design a SP location mechanism which approximate the median or the center locations. Those two problems have been treated in the papers [1, 2, 14, 16, 18, 19, 25]. For trees, cycles, and all networks approximate SP location mechanisms based on randomization were designed in [1, 2].

The main objective of the PhD thesis is to investigate the strategyproofness of mechanisms based on median points in main classes of graphs investigated in metric graph theory, which can be viewed as a far-reaching generalization of trees. The second objective is to investigate approximability of SP location mechanisms with respect to median or barycenter functions in graphs.

2. The model

We use the model of Schummer and Vohra [25]. Let $N = \{1, \ldots, n\}$ be the set of agents. The metric space is an undirected unweighted connected graph $G = (V, E)$ endowed with the standard graph distance $d := d_G$. Each agent $i \in N$ has a *location* x_i, which is a vertex of G. The collection $\pi = (x_1, \ldots, x_n)$ is referred to as the *location profile* (or simply, a *profile*). Denote by V^* the set of all location profiles of finite length.

A *mechanism* (or a *consensus function*) on a graph $G = (V, E)$ is any function $L : V^* \to V$. A *generalized mechanism* (or a *generalized consensus function*) on a graph $G = (V, E)$ is any function $L : V^* \to 2^V \setminus \varnothing$. When the facility is located at vertex $v \in V$ (i.e. $L(\pi) = v$), then the cost of agent i is the distance $d(x_i, v)$ from x_i to v. More generally, if L is a generalized mechanism, then the cost of agent i is the distance $d(x_i, L(\pi))$ from x_i to the set $L(\pi)$ (recall that the distance from a vertex x to a set A is the smallest distance $d(x, a)$ for $a \in A$).

Given a profile $\pi = (x_1, \ldots, x_n)$ and a vertex v of G, let

$$F_\pi(v) = \sum_{i=1}^{n} d(v, x_i).$$

A vertex v of G minimizing $F_\pi(v)$ is called a *median vertex* of G with respect to π and the set of all medians vertices is called the *median set* $\mathrm{Med}(\pi)$. Finally, the mapping Med that associate to any profile π of G its median set $\mathrm{Med}(\pi)$ is called the *median mechanism*. Clearly, Med is a generalized mechanism. A mechanism $L : V^* \to V$ is called a *simple median mechanism* if for any profile $\pi \in V^*$, $L(\pi) \in \mathrm{Med}(\pi)$. We will denote any simple median mechanism by Med^0.

Analogously, for a profile π and a vertex v of G, let

$$B_\pi(v) = \sum_{i=1}^{n} d^2(v, x_i).$$

A vertex v of G minimizing $B_\pi(v)$ is called a *barycenter vertex* of G with respect to π and the set of all barycenter vertices is called the *barycenter set* $\mathrm{Bary}(\pi)$. The *barycenter mechanism* Bary and the simple center mechanism Bary^0 are defined analogously to Med and Med^0.

We say that a mechanism $L : V^* \to V$ is an *α-approximation* of the simple median mechanism Med^0 (or of the simple barycenter mechanism Bary^0) if for any profile $\pi \in V^*$, $F_\pi(L(\pi)) \leq \alpha \cdot F_\pi(\mathrm{Med}^0(\pi))$ (respectively, $B_\pi(L(\pi)) \leq \alpha \cdot B_\pi(\mathrm{Bary}^0(\pi))$).

Now, we continue with the definition of strategyproof location mechanisms. A mechanism $L : V^* \to V$ (or a generalized mechanism $L : V^* \to 2^V$) is called *strategyproof* if for any profile $\pi = (x_1, \ldots, x_n) \in V^*$, any agent $i \in N$, and any vertex $x'_i \neq x_i$ of G, we have

$$d(x_i, L(\pi - \{x_i\} \cup \{x'_i\})) \geq d(x_i, L(\pi)).$$

Consequently, this axiom says that no agent i can improve his distance to the facility location $L(\pi)$ by misreporting his position, i.e., reporting x'_i instead of x_i. More generally, a mechanism $L : V^* \to V$ is called *group strategyproof* if for any profile $\pi = (x_1, \ldots, x_n) \in V^*$, any subset $i_1, \ldots, i_k \in N$ of agents, and any vertices $x'_{i_1} \ldots x'_{i_k}$, there exists $i \in \{i_1, \ldots, i_k\}$ such that

$$d(x_i, L(\pi - \{x_{i_1}, \ldots, x_{i_k}\} \cup \{x'_{i_1}, \ldots, x'_{i_k}\})) \geq d(x_i, L(\pi)).$$

This axiom says that if a group of agents lie on their positions, then at least one of the lying agents will not benefit from lying.

3. The subject

We continue with the formulation of main objectives of the PhD thesis.

3.1. **Strategyproofness in median graphs.** Median graphs are the most important class of graphs in metric graph theory. They occur and are important in geometric group theory, universal algebra, concurrency, complexity, and social group choice. For various characterizations of median graphs, see the survey [7]. A graph $G = (V, E)$ is *median* if any triplet of vertices u, v, w of G has a unique median m, i.e., a vertex laying simultaneously on (u, v)-, (v, w)-, and (w, u)-shortest paths.

The median problem in median graphs is well-studied and well-understood. First, the median set $\mathrm{Med}(\pi)$ of any profile can be computed by the majority rule and local medians are global [5, 26]. Moreover, $\mathrm{Med}(\pi)$ can be computed in linear time $O(|E|)$ [10]. A nice axiomatic characterization of Med in median graphs was given in [22].

The first goal will be to investigate the strategyproofness of the median mechanism Med on median graphs and to establish if median graphs admit simple median mechanisms Med^0. We strongly believe that Med is strategyproof and that the simple median mechanisms Med^0 which assign to any profile π the vertex of $\mathrm{Med}(\pi)$ closest to some basepoint b is strategyproof. This closest vertex is well defined because the median sets $\mathrm{Med}(\pi)$ are gated.

On the other hand, one can show that in some median graphs, no simple median mechanism is group strategyproof. It will be interesting to determine if median graphs admit generalized median mechanisms that are strategyproof. Another interesting direction would be to characterize the median graphs for which some simple median mechanism is group strategyproof. In particular, is this the case for cube-free median graphs? More generally, it will be interesting to design an approximate group strategyproof mechanism for the simple median mechanism on all median graphs. The same questions can be raised for barycenter and simple barycenter mechanisms Bary and Bary^0 on median graphs.

3.2. **Strategyproofness in graphs with connected medians and G^p-connected medians.** The *graphs with connected medians* are the graphs in which $\mathrm{Med}(\pi)$ induces a connected subgraph of G for any profile π. It was shown in [6] that they are exactly the graphs in which any local minimum of the median function F_π is a global minimum. In [6], graphs with connected medians have been characterized in several ways and it was shown that many classes of graphs have connected medians. In particular, the *Helly graphs* (the graphs in which the balls satisfy the Helly property), the *basis graphs of matroids*, and *weakly median graphs*. In [11], the results of [6] have been extended to graphs with G^2-*connected medians* (any $\mathrm{Med}(\pi)$ induces a connected subgraph of G^2, the square of G). It was shown that several other important classes of graphs like *bridged graphs* (and thus *chordal graphs*), *bipartite Helly graphs*, *benzenoids*, etc., are graphs with G^2-connected medians. In [12], the results of [6] and [11] have been used to characterize axiomatically the median function Med of graphs with connected medians and bipartite Helly graphs, thus generalizing the result of [22] for median graphs.

It will be important and interesting to investigate for which of these classes of graphs with connected or G^2-connected medians, the median mechanisms Med or some Med^0 are strategyproof. More generally, characterize all graphs for which Med or some Med^0 is strategyproof. For each class of graphs for which this is not the case, it will be interesting to investigate how to design an approximate strategyproof mechanism for Med^0.

One interesting class is that of basis graphs of matroids. For graphic matroids, this is just the graphs of all spanning trees. It was shown in [6] that a median vertex v of any profile π in a basis graph of a matroid can be easily computed by the greedy algorithm. This provides us with a simple median mechanism. Is this mechanism strategyproof? If not, can we approximate it by a strategyproof mechanism? This is interesting already for graphic matroids.

Another interesting particular case is that of Helly graphs and of bipartite Helly graphs. This is because on one hand, Helly graphs are discrete analogs of injective metric spaces and also because particular Helly graphs can be derived as thickening of median graphs. Note also that bipartite Helly graphs are the graphs in which the medians are Condorcet points [4]. Many SP location results in networks were established assuming that these networks are geodesic metric spaces. Other results were established in Euclidean spaces or spaces with norm. Injective metric spaces represent a far-reaching generalization of tree-networks and ℓ_∞-normed spaces. Finally, Helly graphs, bipartite Helly graphs, and injective metric spaces are universal in the sense that any graph, bipartite graph, metric space embeds into the smallest Helly graph, bipartite Helly graph, injective metric space, respectively.

We also hope that the results for median graphs will be possible to generalize to weakly median graphs. Similarly to median graphs, they have rich structural properties (halfspaces, decomposition into primes, etc). Last but not least, it will be interesting to investigate approximate SP location mechanisms for other graphs which can be isometrically embeddable into hypercubes, generalizing median graphs. Here we can mention *ample partial cubes*, *cellular and hypercellular graphs*, benzenoids. It will be interesting to investigate which classes of those graphs lead to *Condorcet domains*. For definitions, see [7].

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